Week 11: System Analysis in the Frequency Domain (Part II)

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Sinusoidal Transfer Function

- **For sinusoidal inputs**

$$G(j\omega) = |G(j\omega)|e^{j\phi}$$

$$|G(j\omega)| = \left|\frac{Y(j\omega)}{U(j\omega)}\right| = \text{amplitude ratio of the output sinuisoid to the input sinusoid}$$

$$\angle G(j\omega) = \angle \frac{Y(j\omega)}{U(j\omega)} = \text{phase shift of the output sinusoid with respect to the input sinusoid}$$

- **First Order System**

$$|G(j\omega)| = \frac{K}{\sqrt{1+\tau^2\omega^2}}$$

$$\angle G(j\omega) = \varphi = -\arctan(\tau\omega)$$

- **Second Order System**

$$|G(j\omega)| = \sqrt{R^2 + I^2}$$

$$\angle G(j\omega) = \varphi = \begin{cases} \arctan\left(\frac{I}{R}\right) & R > 0 \\ \arctan\left(\frac{I}{R}\right) \pm \pi & R \le 0 \end{cases}$$

Second Order Systems

- **Resonant Frequency**

  - Frequency at which amplitude is a maximum

$$\frac{d}{d\omega}|G(jw)| = 0 \qquad \omega_r = \omega_0\sqrt{1-2\zeta^2} \qquad 0 \le \zeta \le 1/\sqrt{2}$$

  - As the damping ratio approaches zero, the resonant frequency approaches natural frequency.

  - Resonant frequency is lower than damped natural frequency, which is exhibited in the transient response

  - For $0.707 < \zeta$ there is no resonant peak, magnitude decreases monotonically with increasing frequency
  - Step response is oscillatory but well-damped and hardly perceptible

Second Order Systems

- **Magnitude of the Resonant Peak**

$$\omega_r = \omega_0\sqrt{1-2\zeta^2} \qquad 0 \le \zeta \le 1/\sqrt{2}$$

$$M_r = |G(jw)|_{max} = |G(j\omega_r)| = \frac{K}{2\zeta\sqrt{1-\zeta^2}}$$

$$\underline{/G(j\omega)} = \varphi(\omega_r) = -\arctan\left(\frac{\sqrt{1-2\zeta^2}}{\zeta}\right)$$

- As the damping ratio approaches zero, the magnitude approaches infinity.

Bode Form of the Transfer Function

- Components of transfer functions

1. $constants$ (gain)
2. $(j\omega)^n$
3. $(j\omega\tau + 1)^{\pm 1}$
4. $\left[\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1\right]^{\pm 1}$

- Break points (corner frequency)

2. $\omega_b = 1/\tau$
3. $\omega_b = \omega_0$

Lecture Overview

- Higher order transfer functions and Delay
- Magnitude in Decibels
- Filter design

Bode Form of the Transfer Function

- Manipulate transfer function into Bode form
- Extend the low frequency asymptote until the first break point. Then step the slope by ±1 or ±2, depending on whether the break point is from a first order or second order term in the numerator or denominator
- Continue through all break points in ascending order
- Approximate magnitude curve is increased from the asymptote value by a factor of 1.4 at first order numerator break point and decreased by a factor of 0.707 at first-order denominator break points
- At second-order break points, find resonant peak

- Plot low frequency asymptote of the phase curve φ = n x 90°
- Approximate phase curve changes by ±90° or ±180° at each break point in ascending order. Sign depends on location.
- Graphically add each phase curve

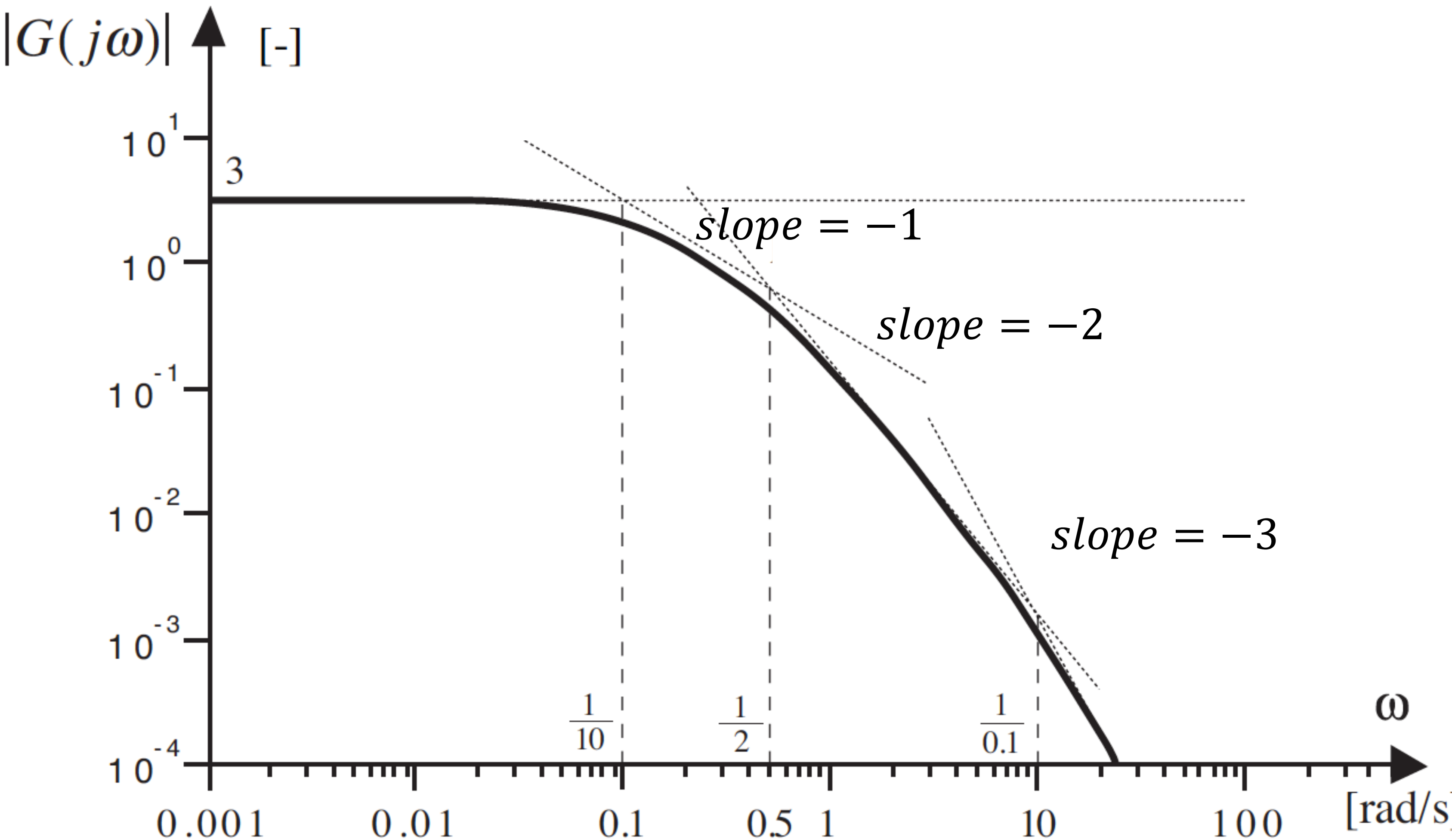
Higher Order Systems

$$G(s) = \frac{3}{(10s+1)(2s+1)(0{,}1s+1)}$$

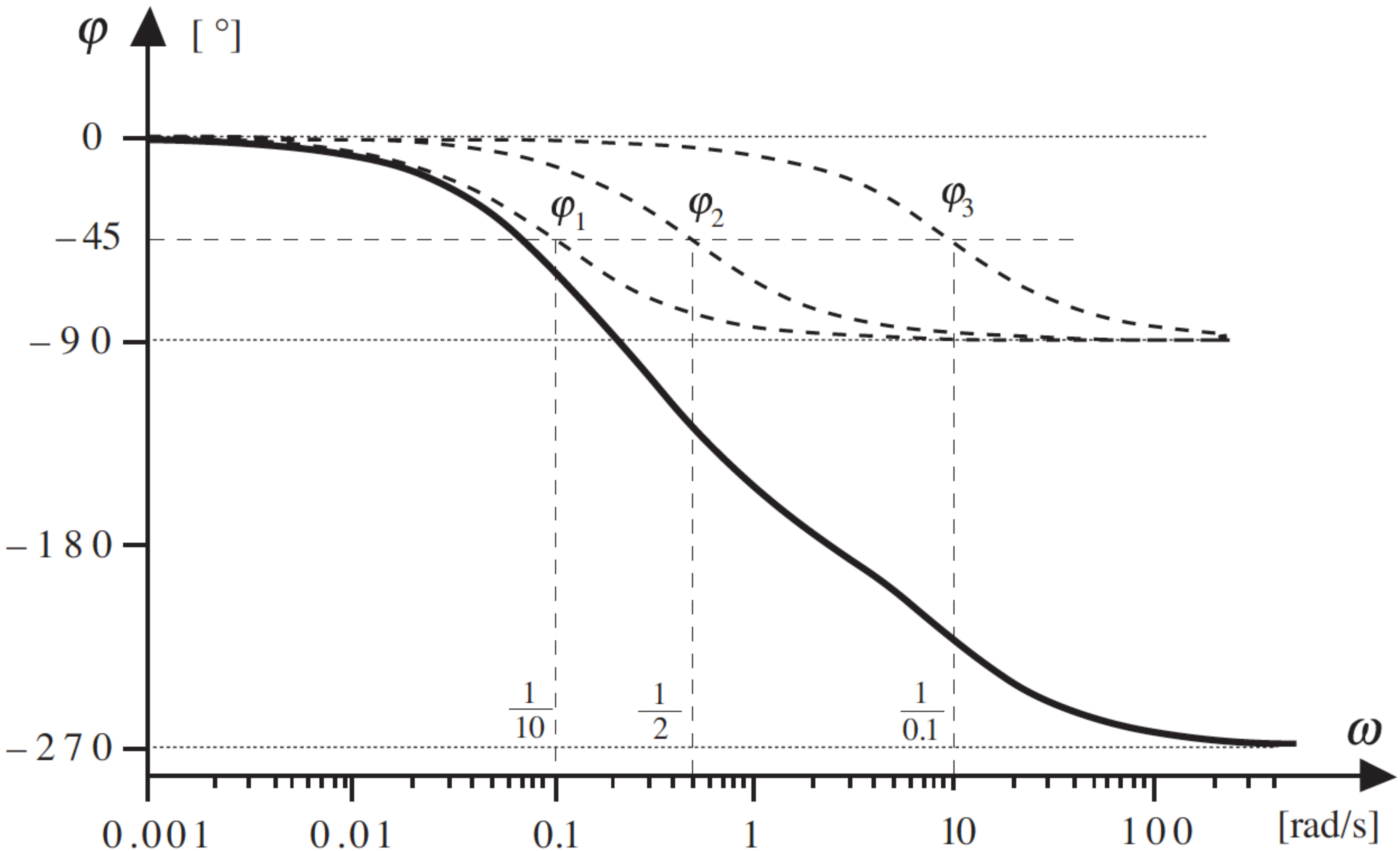
$$|G(j\omega)| = \frac{3}{\sqrt{1+(10\omega)^2}\sqrt{1+(2\omega)^2}\sqrt{1+(0{,}1\omega)^2}}$$

$$\varphi = \arg[G(j\omega)] = -\arctan(10\omega) - \arctan(2\omega) - \arctan(0{,}1\omega)$$

Higher Order Systems

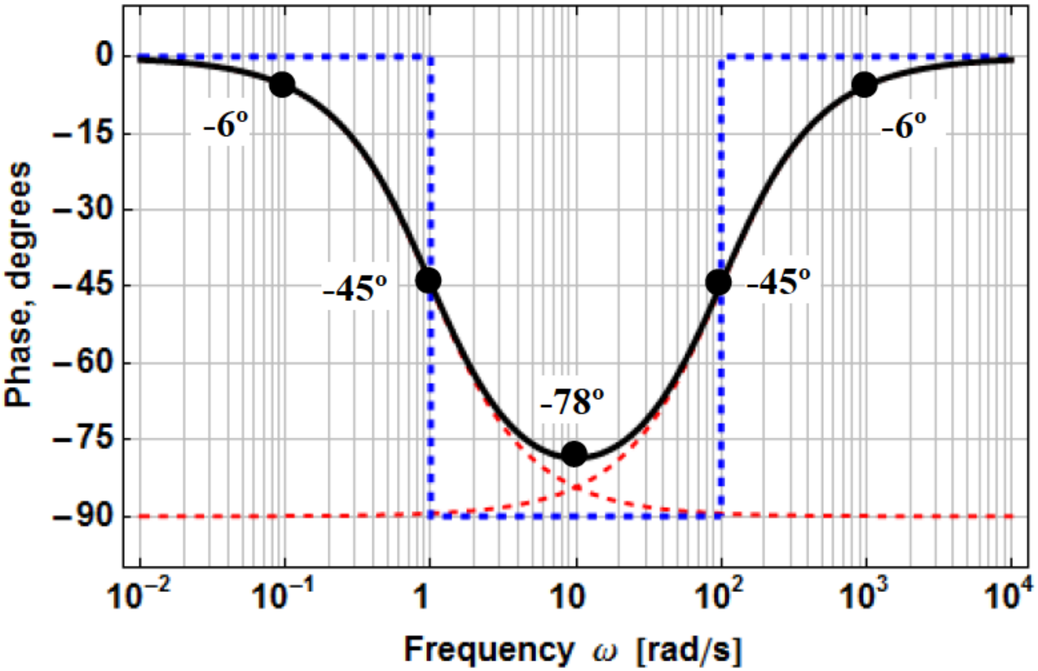


Higher Order Systems



Correction for Phase Angle

$$G(s) = 10\frac{(s+100)}{(s+1)}$$



Systems with Zeros

$$G_1(s) = \frac{2s+1}{5s+1}$$

$$G_2(s) = \frac{-2s+1}{5s+1}$$

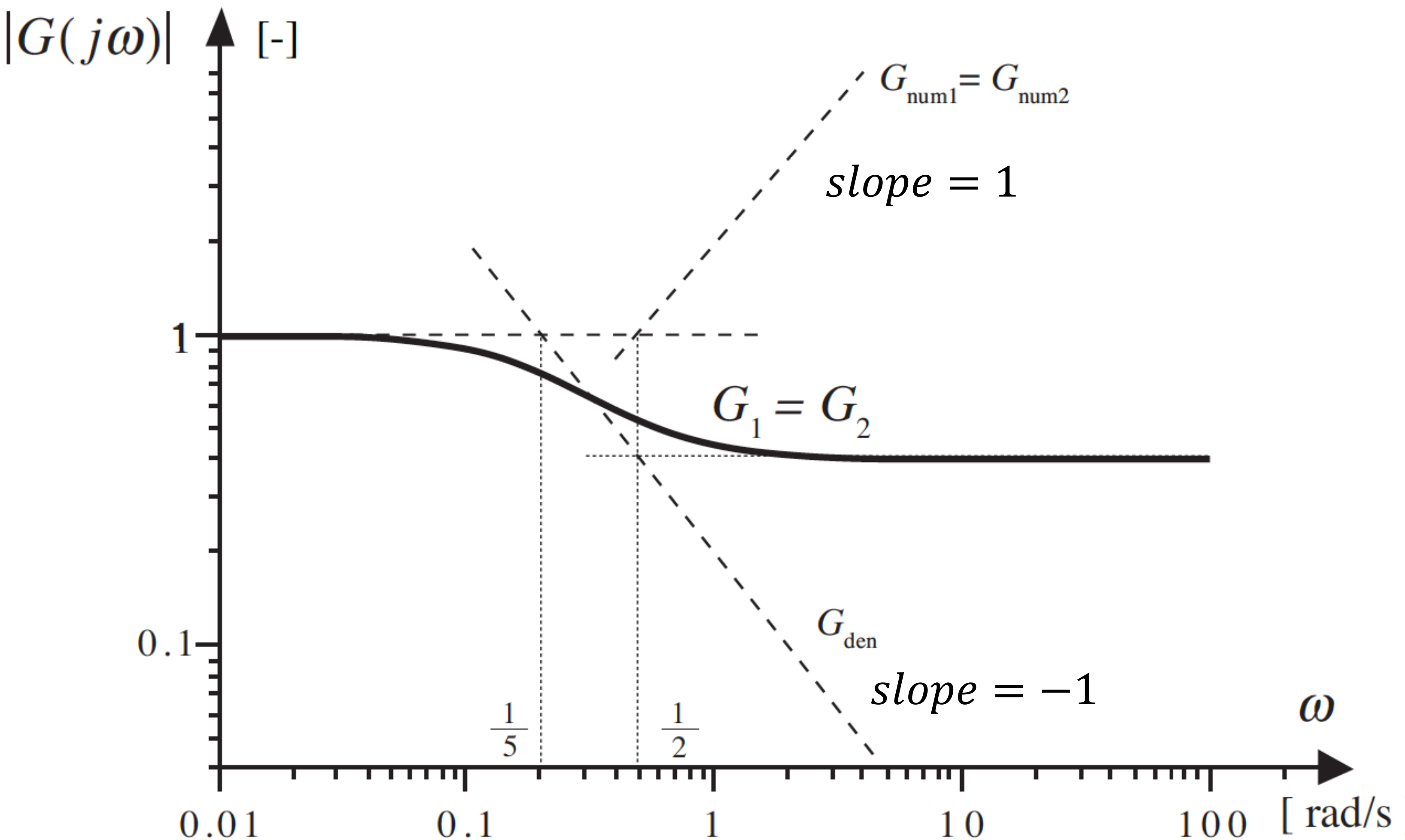
$$|G(j\omega)| = \frac{\sqrt{1+(2\omega)^2}}{\sqrt{1+(5\omega)^2}}$$

$$|G(j\omega)| = \frac{\sqrt{1+(-2\omega)^2}}{\sqrt{1+(5\omega)^2}}$$

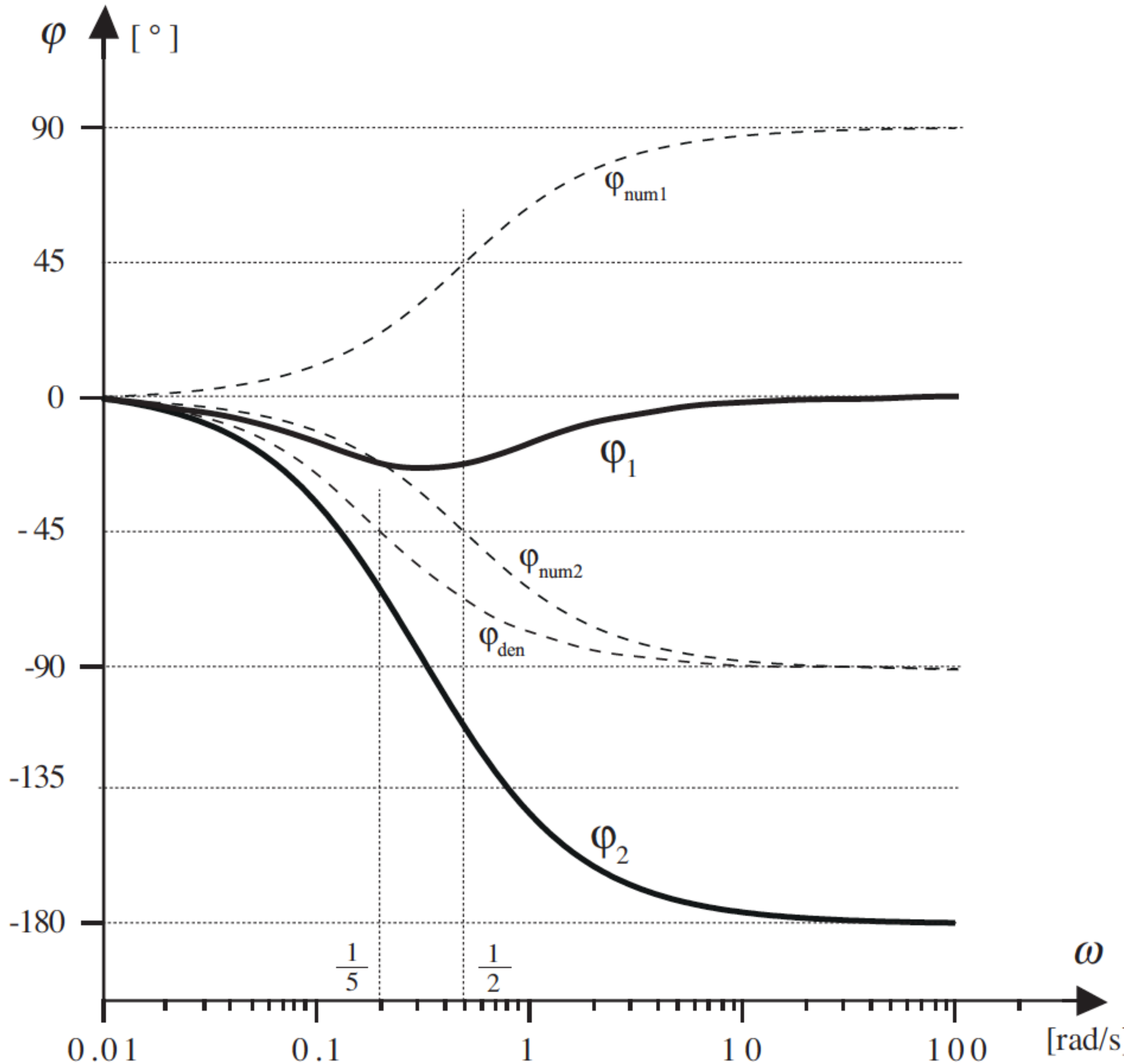
$$\varphi_1 = \arctan(2\omega) - \arctan(5\omega)$$

$$\varphi_2 = -\arctan(2\omega) - \arctan(5\omega)$$

Systems with Zeros



Systems with Zeros



Systems with Zeros

$$G_1(s) = \frac{2s+1}{5s+1}$$

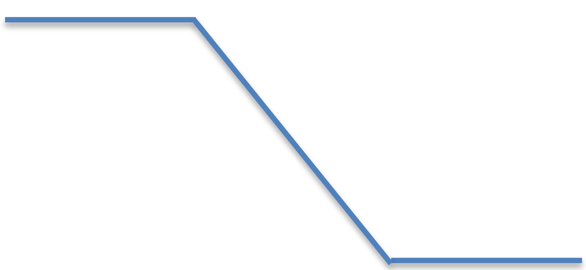
$$G_2(s) = \frac{5s+1}{2s+1}$$

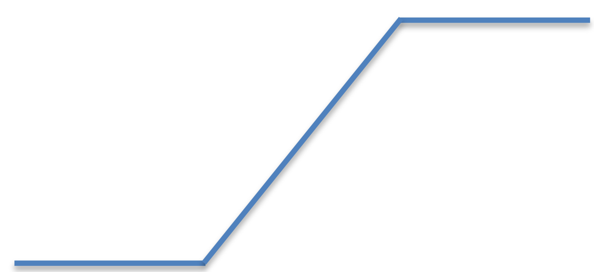
$$|G(j\omega)| = \frac{\sqrt{1+(2\omega)^2}}{\sqrt{1+(5\omega)^2}}$$

$$|G(j\omega)| = \frac{\sqrt{1+(5\omega)^2}}{\sqrt{1+(2\omega)^2}}$$

$$\varphi_1 = \arctan(2\omega) - \arctan(5\omega)$$

$$\varphi_2 = -\arctan(2\omega) + \arctan(5\omega)$$



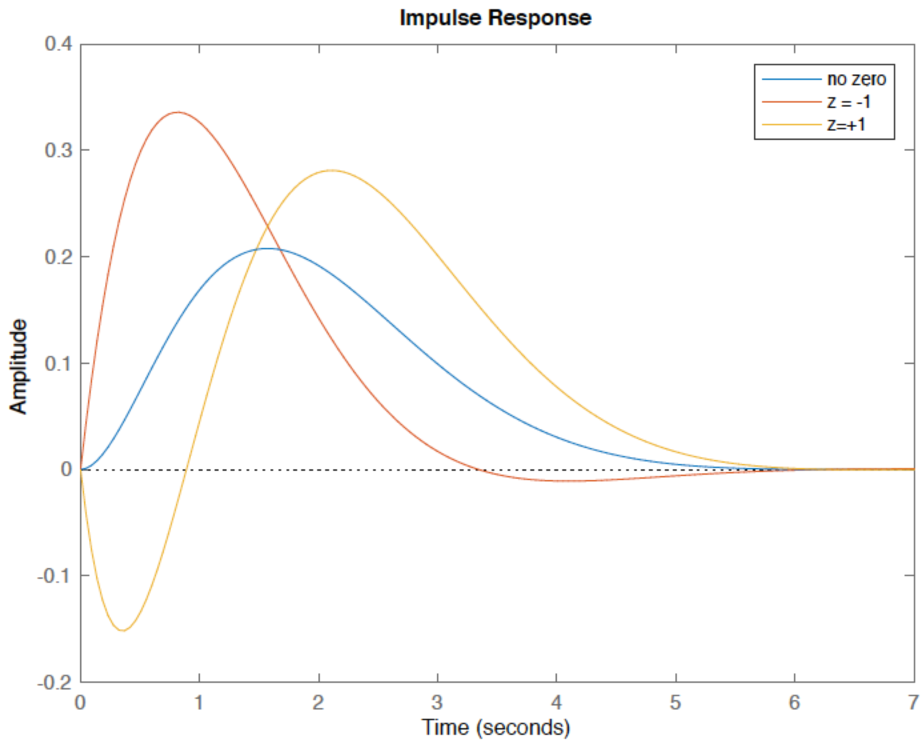


Non-minimum phase zeros

- Minimum phase systems are with
  - Transfer functions having neither poles nor zeros in the right-half s-plane
- Non-minimum phase systems have
  - Transfer functions with poles and/or zeros in the right-half s-plane

- Poles with positive real part result in unstable system (the output diverges over time)
- The stability of the system is preserved when zeros have positive real part
- A zero in the right half plane means a negative derivative action – the output will tend to move in the wrong direction initially
  - Nonminimum phase zeros make the system slow in response because of the faulty behavior at the start of the response
  - Excessive phase lag should be avoided

Non-minimum phase zeros



Systems with Delay

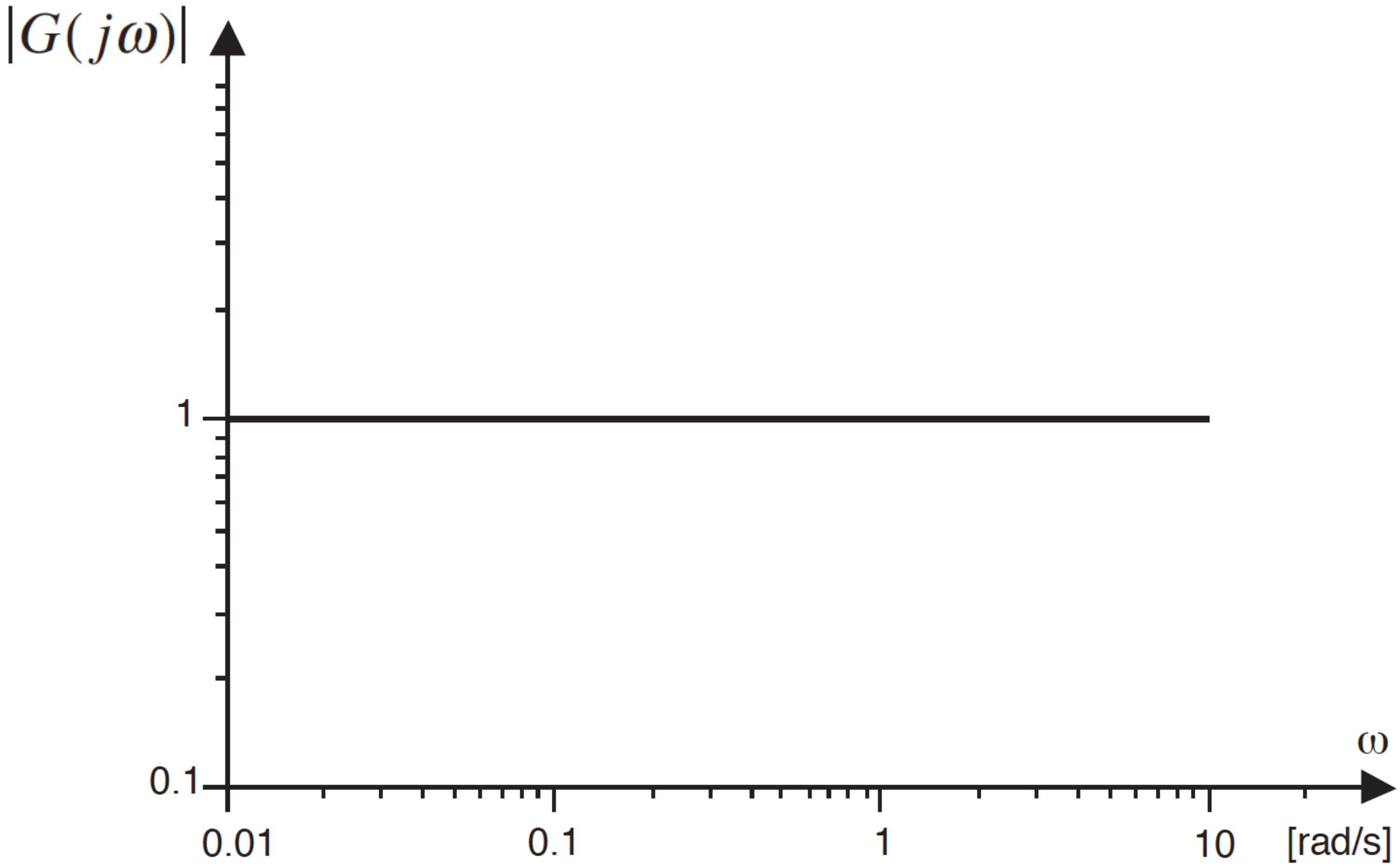
$$y(t) = u(t-\theta) \xrightarrow{\mathcal{L}} Y(s) = \exp(-\theta s)U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \exp(-\theta s)$$

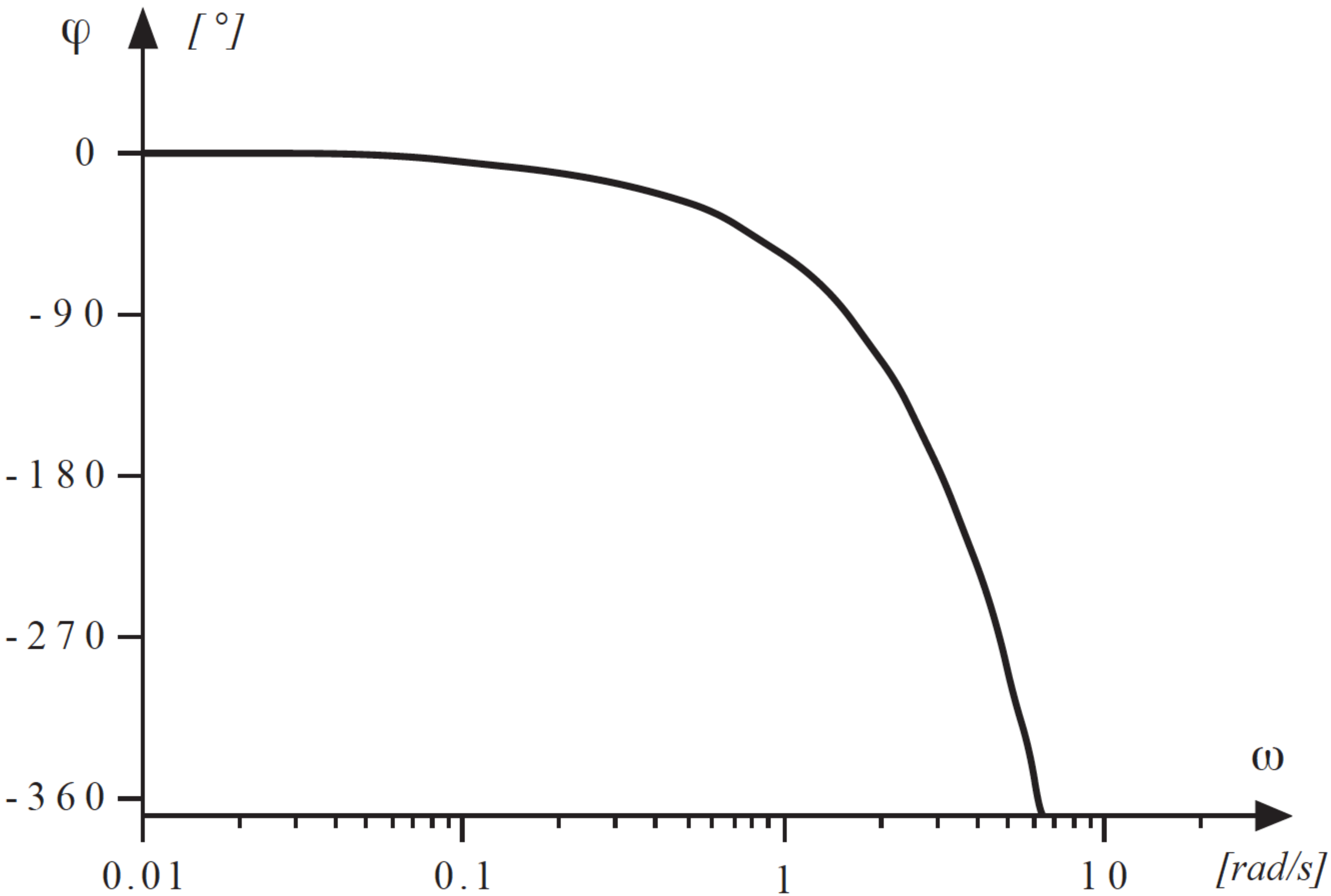
$$|G(j\omega)| = |\exp(-j\theta\omega)| = 1$$

$$\varphi = \arg[\exp(-j\theta\omega)] = -\theta\omega$$

Systems with Delay

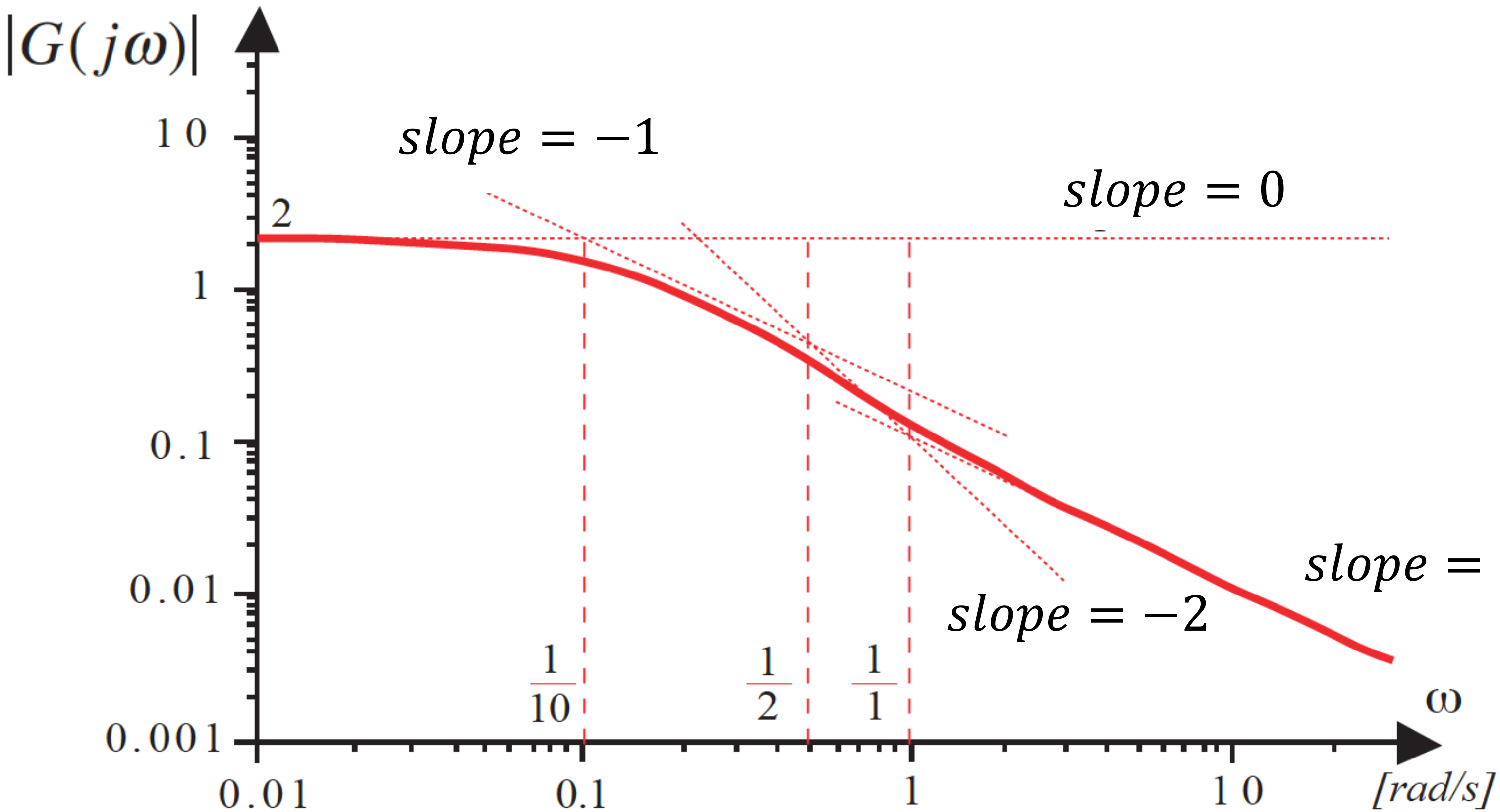


Systems with Delay



Example

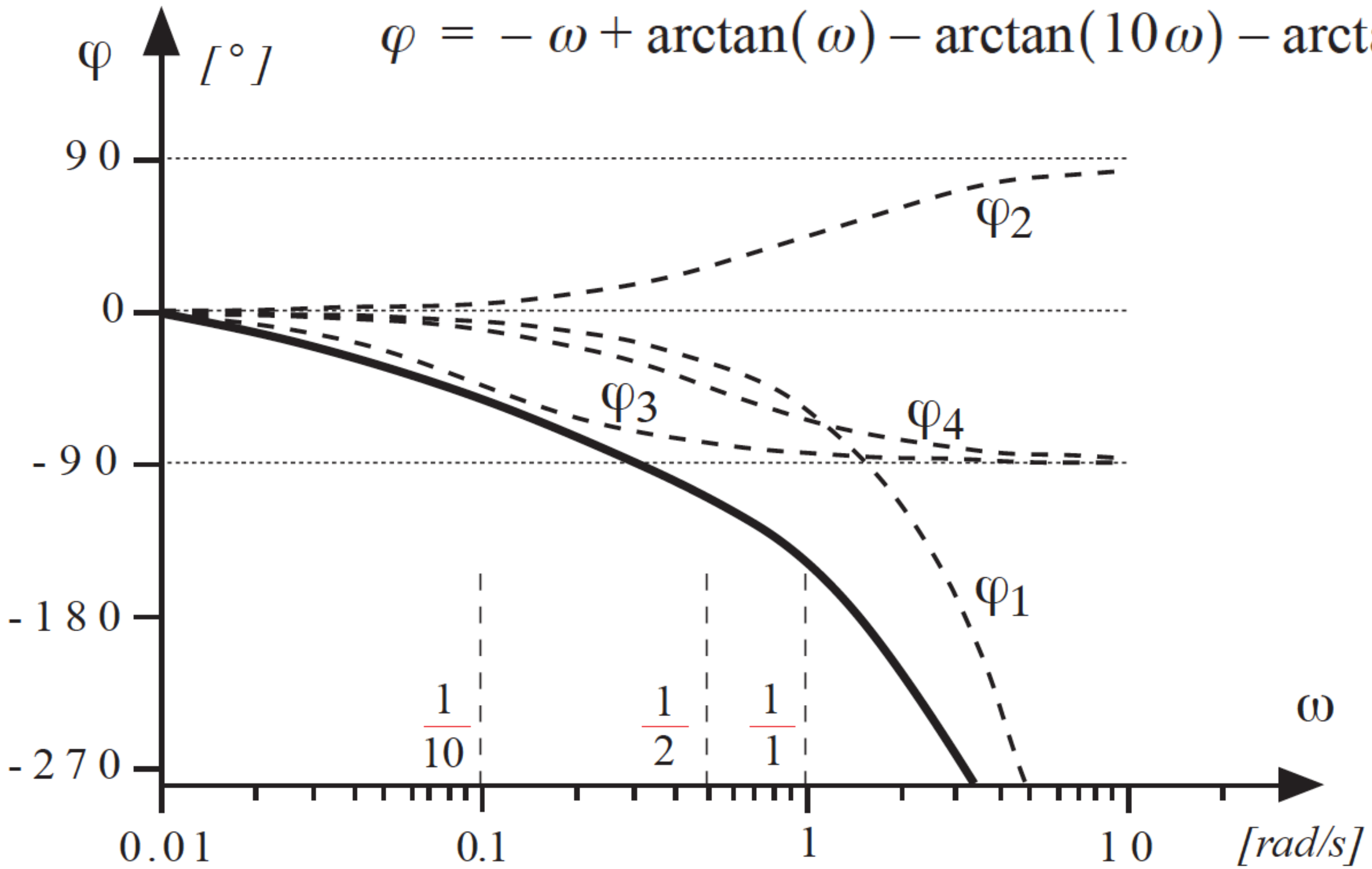
$$G(s) = \frac{2(s+1)\exp(-s)}{(10s+1)(2s+1)} \qquad |G(j\omega)| = 2\frac{\sqrt{1+\omega^2}}{\sqrt{1+(10\omega)^2}\sqrt{1+(2\omega)^2}}$$



Systems with Delay

$$G(s) = \frac{2(s+1)\exp(-s)}{(10s+1)(2s+1)}$$

$$\varphi = -\omega + \arctan(\omega) - \arctan(10\omega) - \arctan(2\omega)$$



Bode Form of the Transfer Function

- Components of transfer functions

1. $constants$ (gain)
2. $(j\omega)^n$
3. $(j\omega\tau + 1)^{\pm 1}$
4. $\left[\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1\right]^{\pm 1}$
5. $e^{-ja\omega}$ (delay)

- Break points (corner frequency)

2. $\omega_b = 1/\tau$
3. $\omega_b = \omega_0$

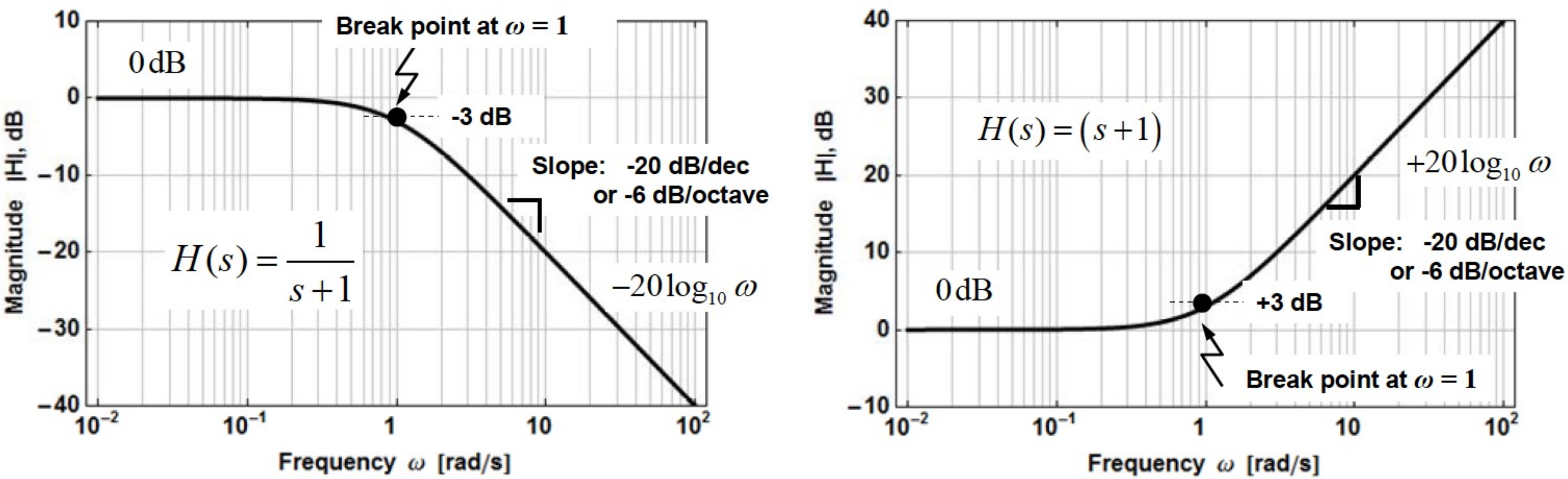
- Bandwidth and Cut-off frequency

Decibel (power dB)

- In communications, it is standard to measure the power gain in decibels (dB)
- Decibels vs log(w) as a semi log plot

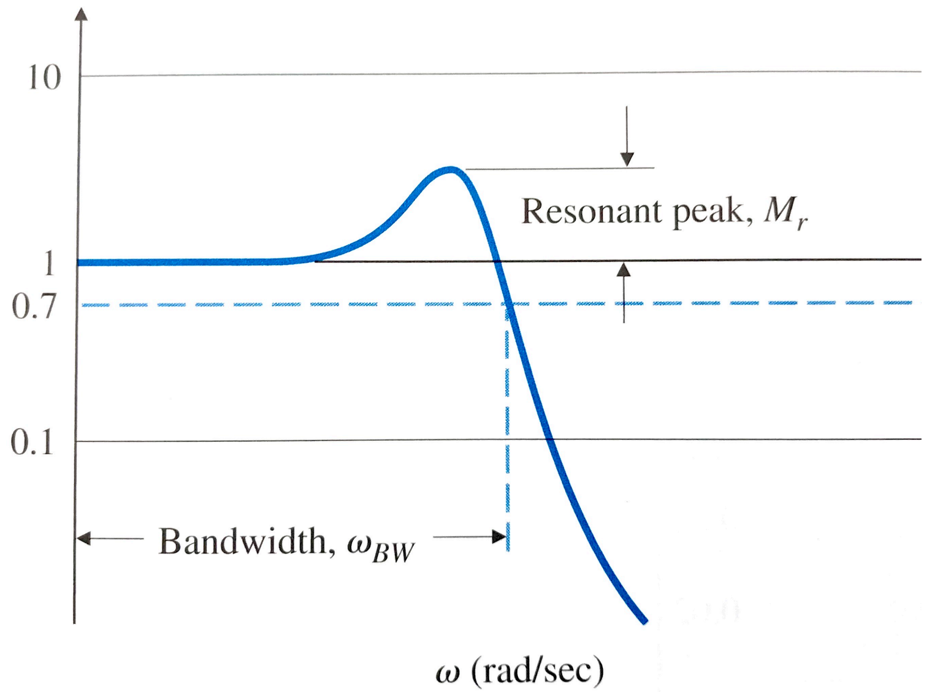
Magnitude in dB

$$|G(j\omega)|_{dB} = 20log_{10}|G(j\omega)|$$



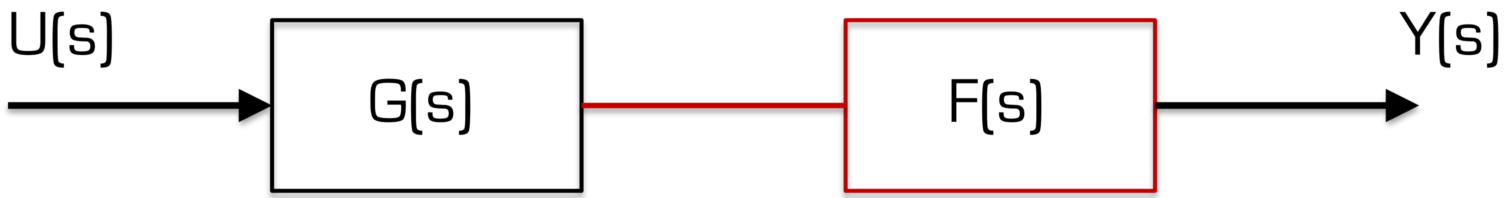
Performance Specifications

- Speed of transient response
  - As bandwidth increases, the rise time of the step response will decrease
  - Bandwidth is proportional to the speed of the response
- As resonant peak increases in magnitude, the percent overshoot increases



Filter Design

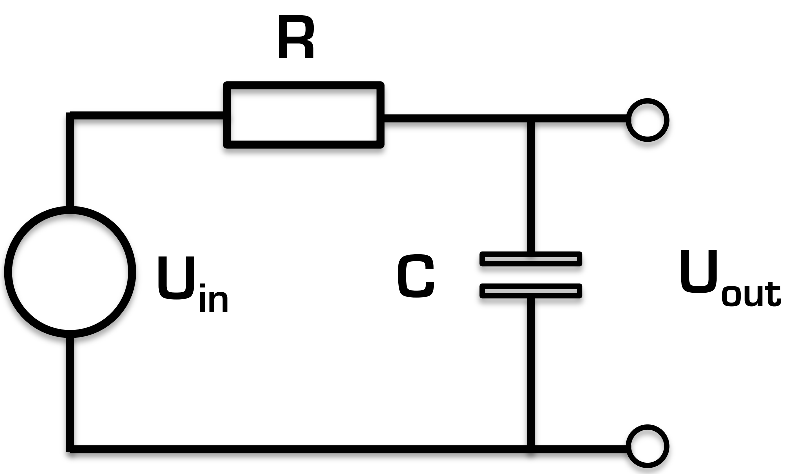
- The objective is to modify certain characteristics of system response
  - Magnitude and phase at a certain frequency
  - **Low-pass filter:** cut unwanted high-frequency components
  - **High-pass filter:** cut unwanted low-frequency components
  - **Band-pass** and **notch filter**: attenuate specific frequencies
  - **All-pass filter** (phasor effect): only change phase



Low-Pass Filter (or Amplifier)

- Amplifies signals below a cut-off frequency, including DC gain
- ω_H = upper cutoff frequency

$$F(s) = K \frac{\omega_H}{(s + \omega_H)} \qquad \varphi = -\arctan(\frac{\omega}{\omega_H})$$



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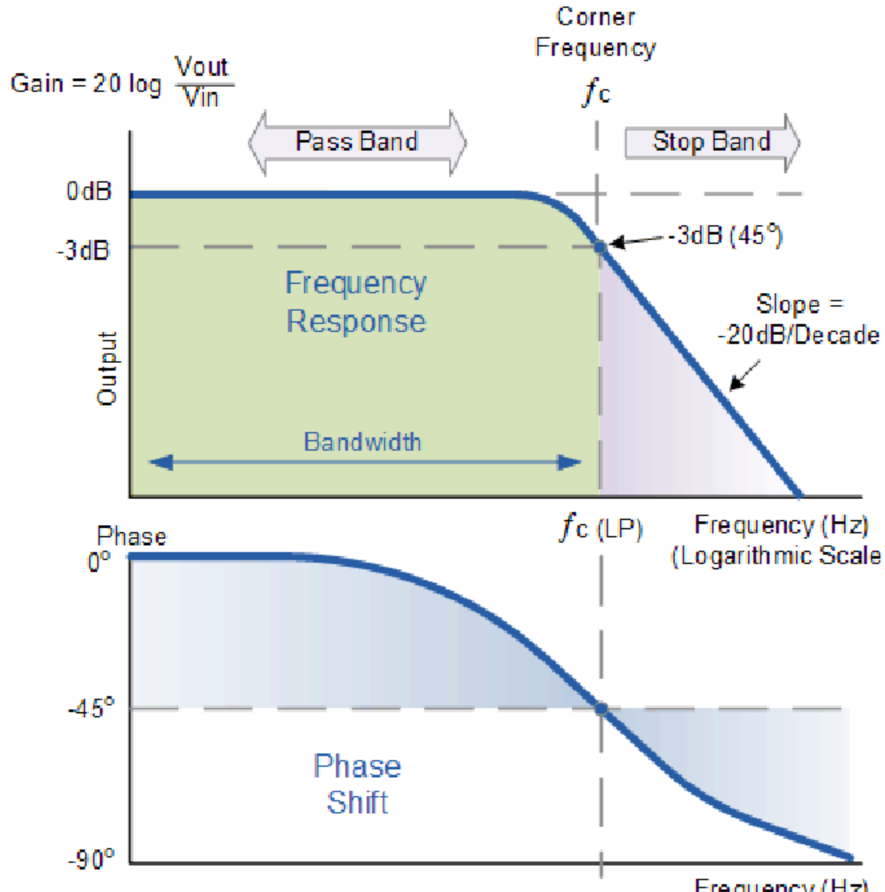
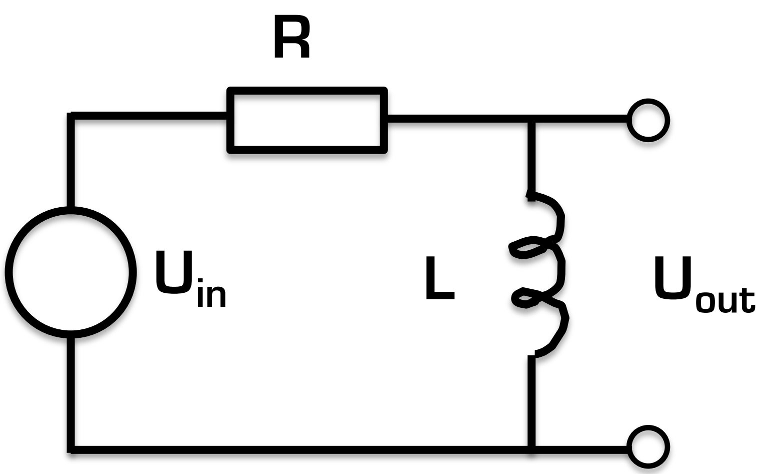


High-Pass Filter (or Amplifier)

- A single pole with a zero at the origin
- ω_L = lower cutoff frequency

$$F(s) = K\frac{s}{(s+\omega_L)} \qquad \varphi = 90° - \arctan(\frac{\omega}{\omega_L})$$



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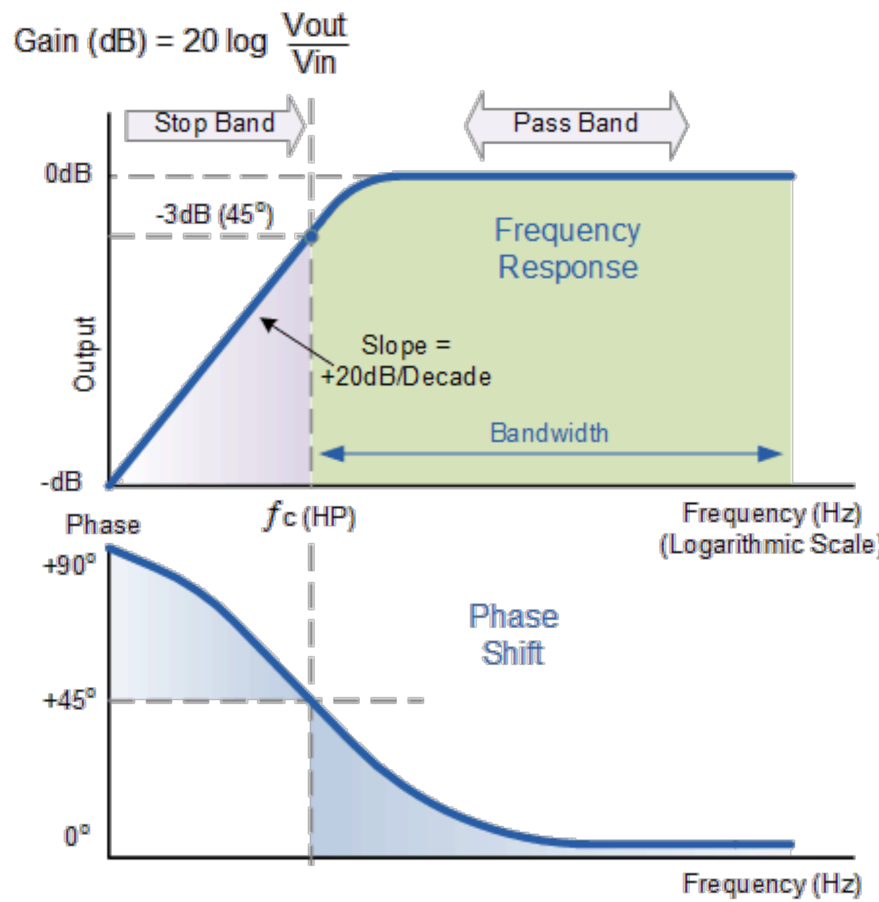


High-Q Band-Pass Amplifier and Notch Filter

- Band-pass: Combination of high-pass and low-pass characteristics

$$F(s) = K \frac{s\omega_H}{(s+\omega_L)(s+\omega_H)}$$

- For small bandwidth (ω_H - ω_L) and high quality factor (Q), poles must be complex

$$F(s) = K \frac{s\frac{\omega_c}{Q}}{s^2 + s\frac{\omega_c}{Q} + {\omega_c}^2} \qquad \varphi = 90° - \arctan(\frac{1}{Q}\frac{\omega\omega_c}{{\omega_c}^2 - \omega^2})$$

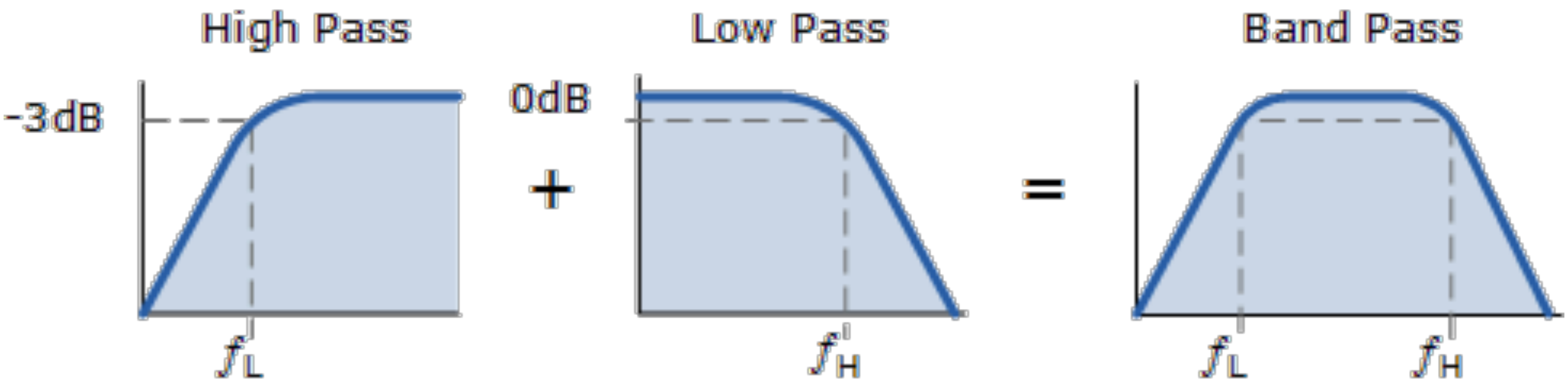
- Band rejection (Notch) filter

$$F(s) = K \frac{s^2 + {\omega_c}^2}{s^2 + s\frac{\omega_c}{Q} + {\omega_c}^2}$$

Band-Pass Filter

- Band-pass: Combination of high-pass and low-pass characteristics

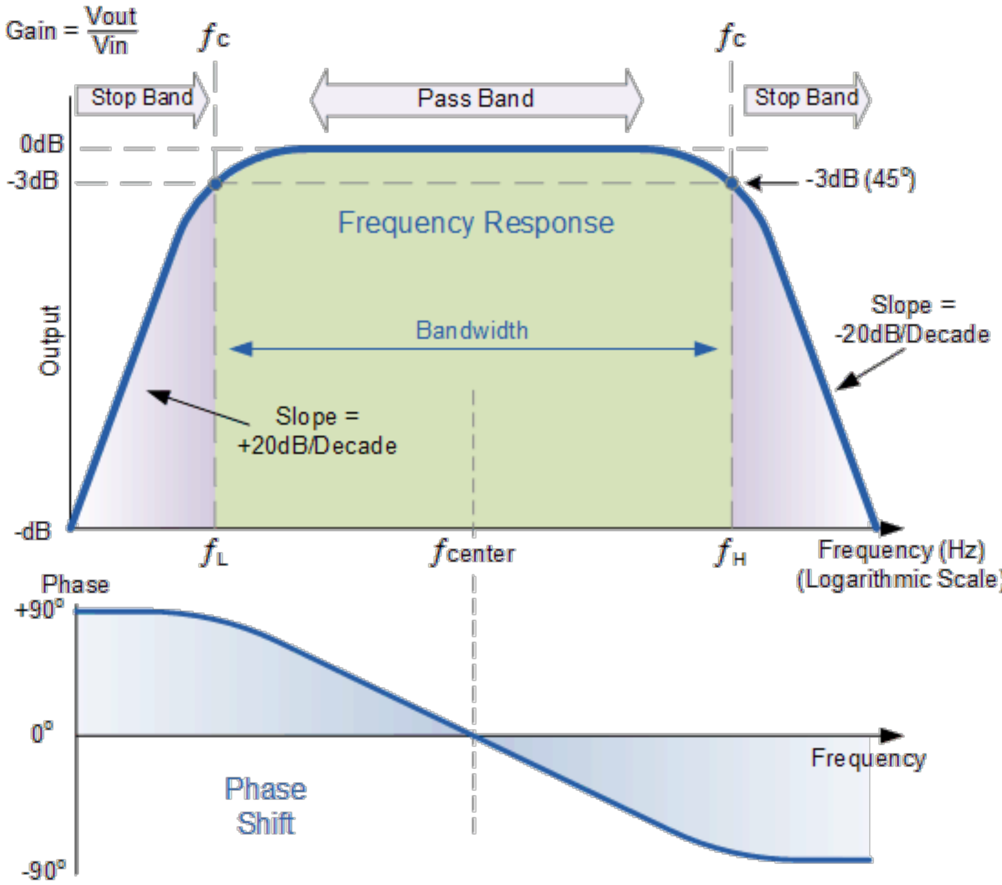
$$F(s) = K \frac{s\omega_H}{(s+\omega_L)(s+\omega_H)}$$



Band-Pass Filter

- Band-pass: Combination of high-pass and low-pass characteristics

$$F(s) = K\frac{s\omega_H}{(s+\omega_L)(s+\omega_H)}$$



All-pass Function

- Uniform magnitude response at all frequencies
- Can be used to tailor phase characteristics of the system

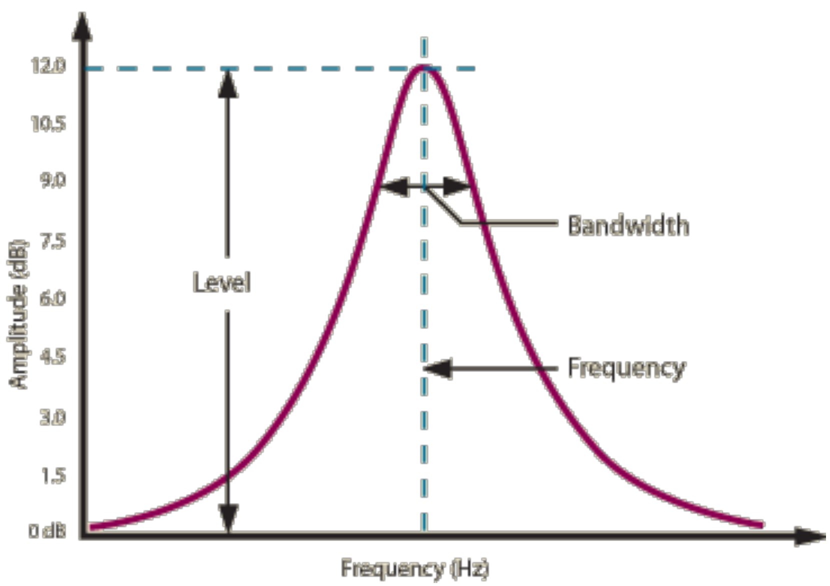
$$F(s) = K\frac{(s-\omega_c)}{(s+\omega_c)}$$

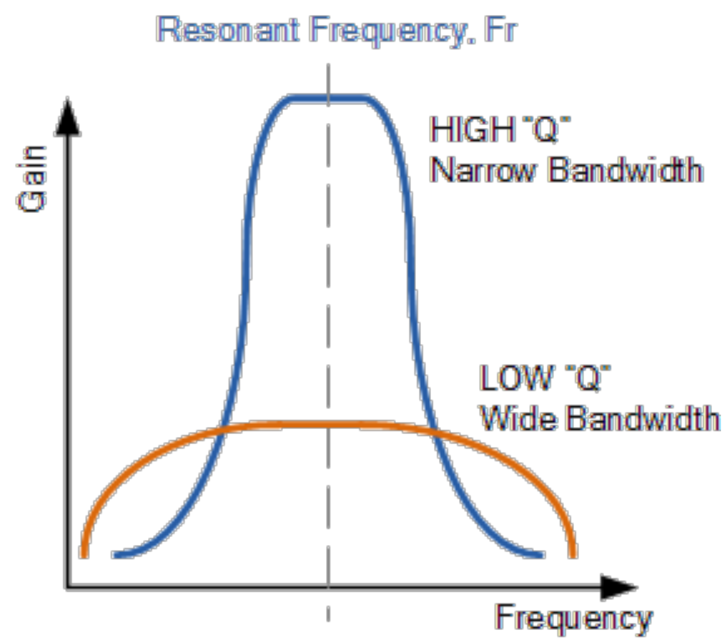
$$\log|G(j\omega)| = \log K$$

$$\varphi = -2arctan\left(\frac{\omega}{\omega_c}\right)$$

Quality Factor

- Dimensionless parameter that describes how underdamped a resonator is
- The higher Q (the "Quality") is, the sharper the resonance is.
- Numerically, the Q-factor is the relation between center frequency and the -3 dB-bandwidth.



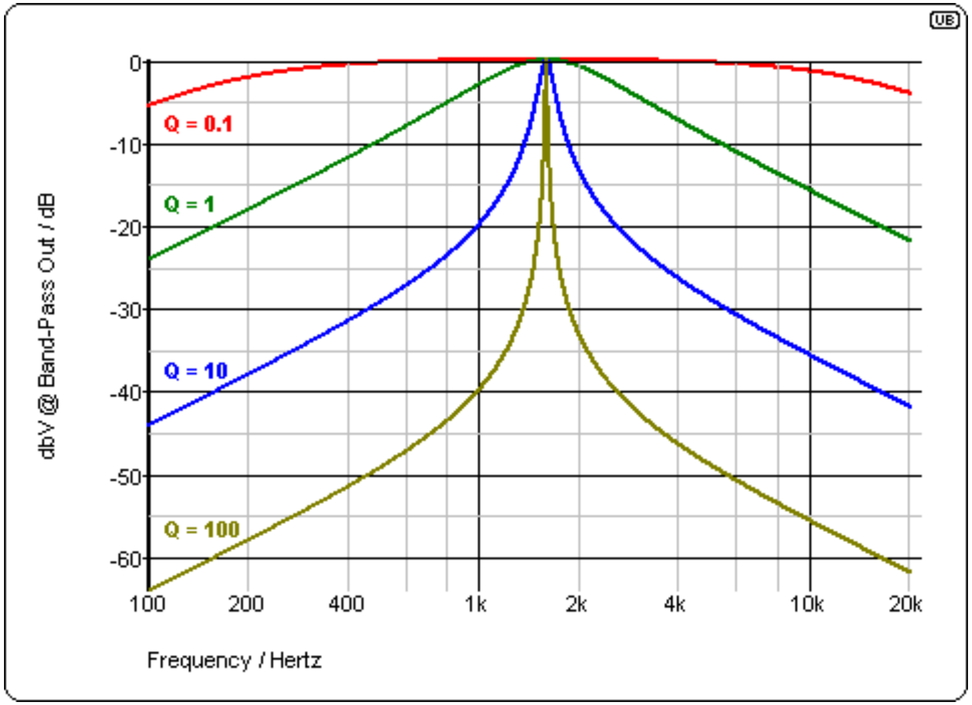


High-Q Band-Pass Filter

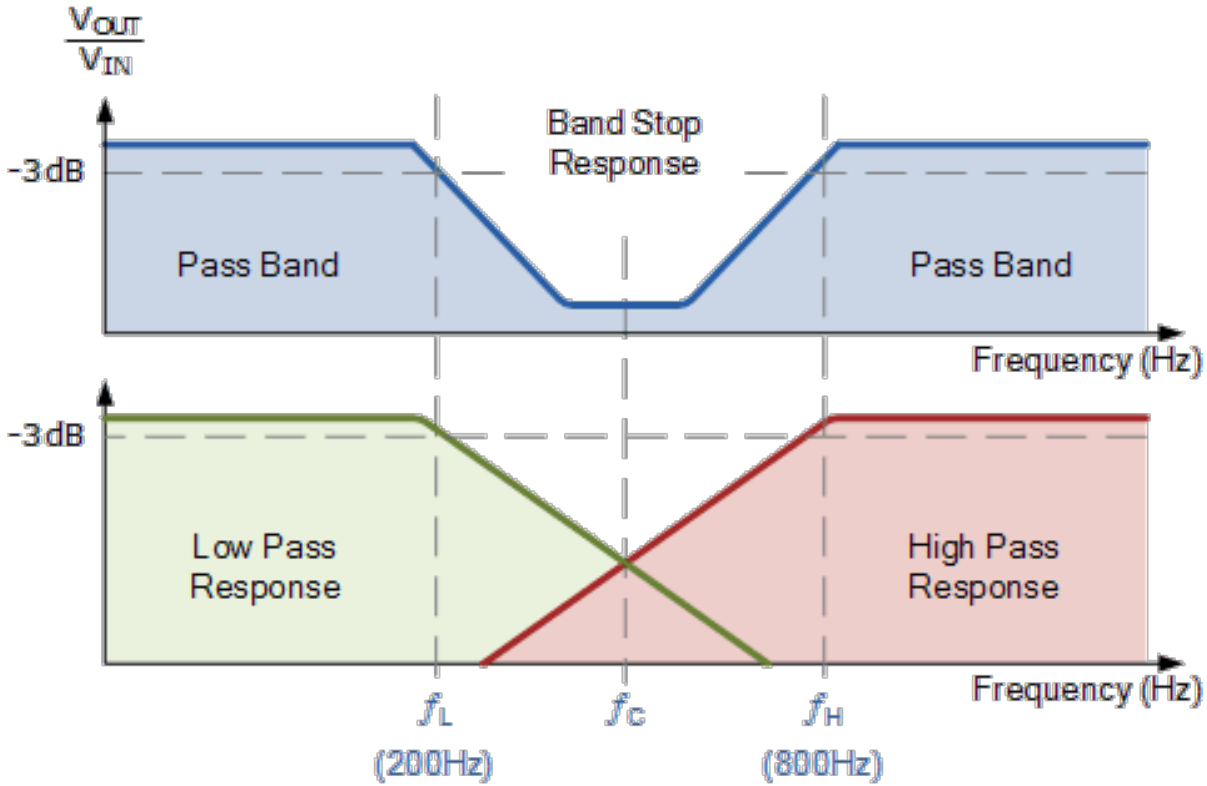
- For small bandwidth (ω_H - ω_L) and high quality factor (Q), poles must be complex

$$F(s) = K \frac{s\frac{\omega_c}{Q}}{s^2 + s\frac{\omega_c}{Q} + {\omega_c}^2}$$

$$\varphi = 90° - \arctan\left(\frac{1}{Q}\frac{\omega\omega_c}{{\omega_c}^2 - \omega^2}\right)$$



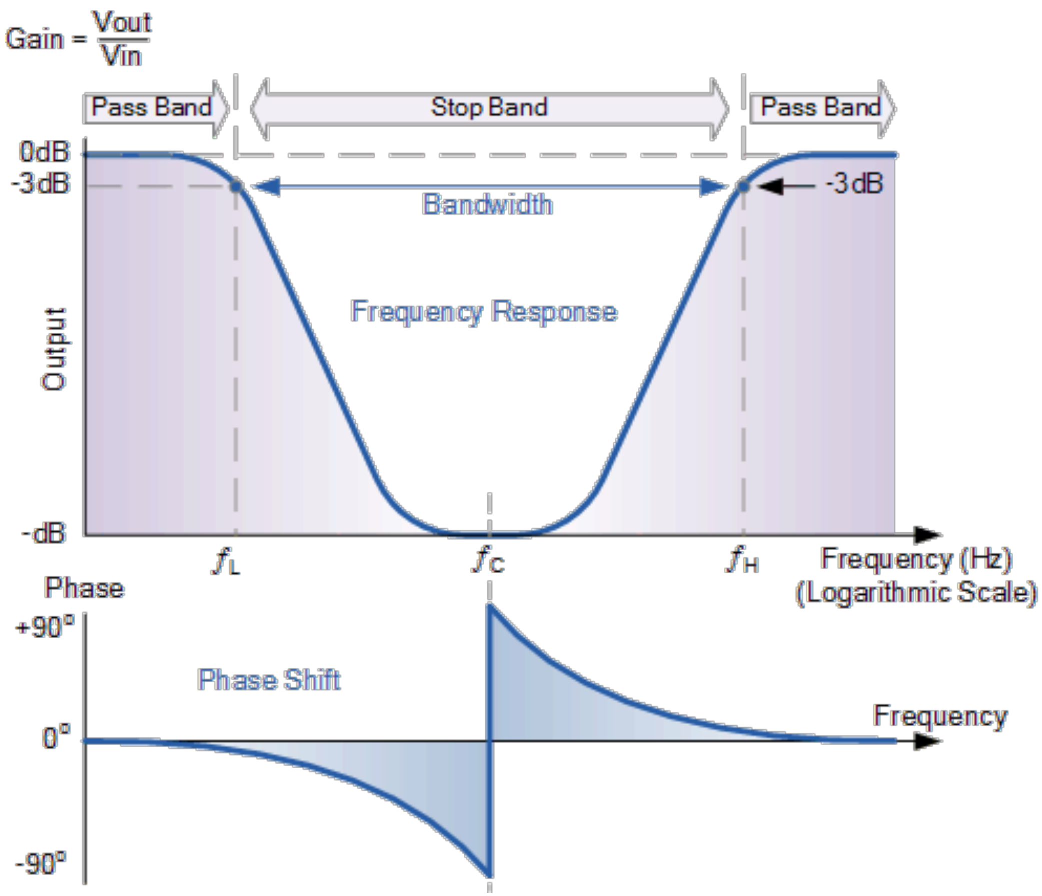
Band-Stop Filter

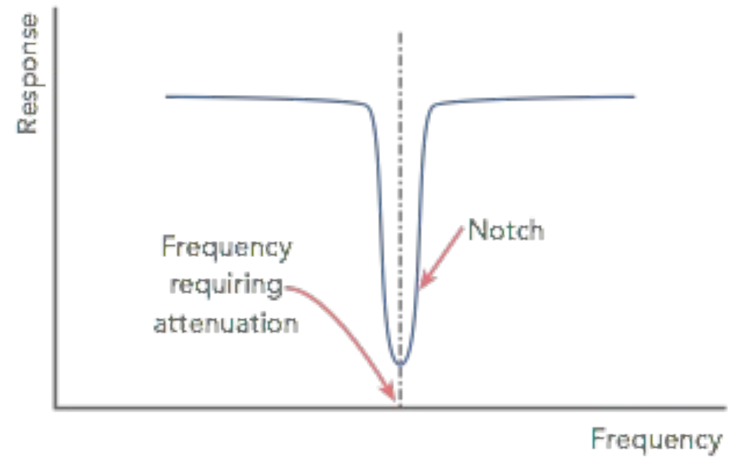


Band-Stop Filter

- Notch filter is high-Q band stop filter

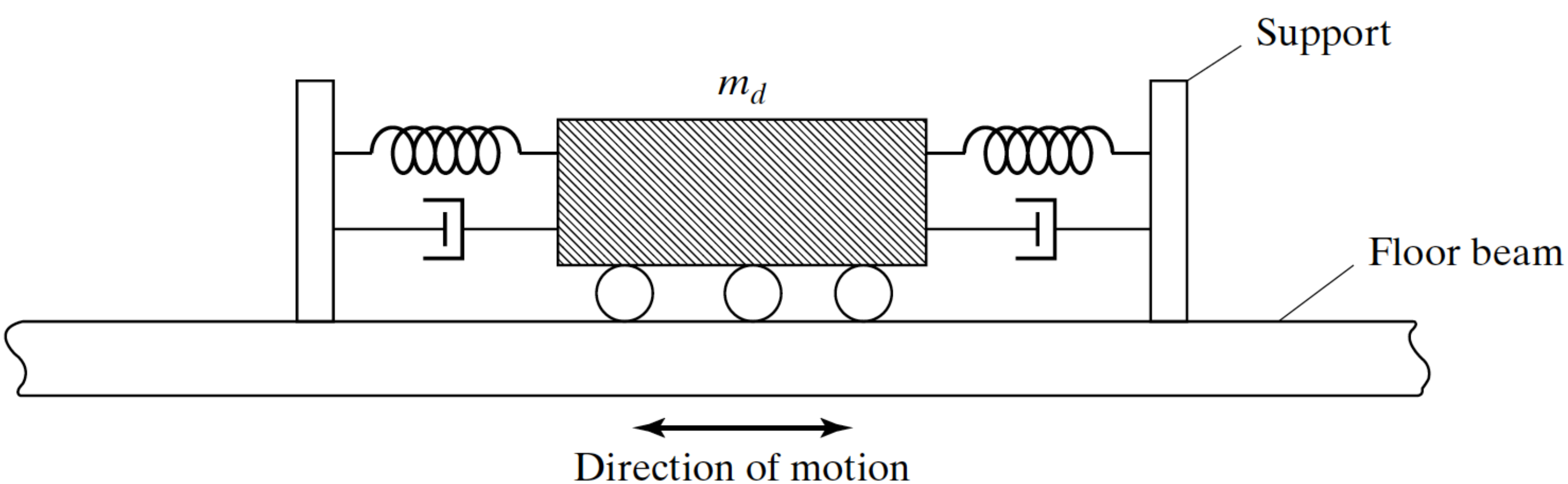
$$F(s) = K\frac{s^2+\omega_c{}^2}{s^2+s\frac{\omega_c}{Q}+\omega_c{}^2}$$





Vibration Absorber

- Tuned spring-mass-damper system which reduces or eliminates the vibration of a harmonically excited system
- Rotating machines
- Tuned to oscillate in such a way that exactly counteracts the force from the rotating imbalance
- Energy dissipation



Example

Consider a mechanical system described by the following differential equation. The system is initially at rest.

$$\ddot{y}(t) + \dot{y}(t) + y(t) = 2u(t)$$

a) Find the transfer function $G(s)$ of the system and sketch the Bode plot.

b) We would like to design a first order filter $F(s) = \dfrac{K}{\tau s + 1}$ in a way that the new system with the transfer function $G'(s) = G(s) \times F(s)$ has magnitude $|G'(j\omega)| = 1$ and phase angle $\phi = -3\pi/4$ at frequency $\omega = 1$.

Example

a) The transfer function can be calculated as:

$$G(s) = \frac{2}{s^2 + s + 1}$$

The second order term has a natural frequency of $\omega_0 = 1\text{rad/sec}$, the damping ratio is $\zeta = 0.5$, and the gain is 2. The resonance frequency is $\omega_r = \omega_0\sqrt{1 - 2\zeta^2} = 0.707$. The resonant peak is $R(\omega_r) = \dfrac{2}{2\zeta\sqrt{1 - 2\zeta^2}} = 2.31$.

Example

b) The filtered system is given by:

$$G'(s) = \frac{K}{\tau s + 1} \frac{2}{s^2 + s + 1}$$

The magnitude of the sinusoidal transfer function at $\omega = 1$ must be 1.

$$|G'(j\omega)| = \frac{2K}{\sqrt{\tau^2\omega^2 + 1}\sqrt{(1-\omega^2)^2 + \omega^2}} \rightarrow |G(\omega = 1)| = \frac{2K}{\sqrt{1+\tau^2}} = 1$$

And the phase angle of the sinusoidal transfer function at $\omega = 1$ must be $-3\pi/4$.

$$\phi(\omega = 1) = -\pi/2 - arctan(\tau) = -3\pi/4$$

As a result, $\tau = 1$ and $K = \sqrt{2}/2 = 0.707$.